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BRIEF COMMUNICATION

Remark on positive entropy of a coupled lattice system related with Belusov–Zhabotinskii reaction

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Abstract In García Guirao and Lampart (J Math Chem 48:159–164, 2010) presented a lattice dynamical system (LDS) stated by Kaneko (Phys Rev Lett 65:1391–1394, 1990) which is related to the Belusov–Zhabotinskii reaction. In this paper, we consider the following more general LDS:

$$x_n^{m+1} = (1-\varepsilon) f_n\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f_n\left(x_{n-1}^m\right) - f_n\left(x_{n+1}^m\right)\right],$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in I = [0, 1]$ is coupling constant and f_n is a continuous selfmap on I for every $n \in \{1, 2, \ldots, L\}$. In particular, we prove that for zero coupling constant, if there is $n \in \{1, 2, \ldots, L\}$ such that f_n has positive topological entropy, then so does this coupled map lattice system. This result extends the existing one.

Keywords Coupled map lattice · Positive topological entropy · Devaney's chaos · Chaos in the sense of Li–Yorke · Tent map

Mathematics Subject Classification: 54H20 · 37B40 · 37D45

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1 Introduction

By a topological dynamical system (TDS) (X, f) we mean a compact metric space X and a continuous map $f : X \to X$. Since Li and Yorke [1] introduced the term of chaos in 1975, TDS were highly considered and studied in the literature (see [2,3]) because they are very good examples of problems coming from the theory of topological dynamics and model many phenomena from biology, physics, chemistry, engineering and social sciences.

Coming from physical/chemical engineering applications, such as digital filtering, imaging and spatial vibrations of the elements which compose a given chemical product, a generalization of classical discrete dynamical systems has recently appeared as an important subject for investigation, we mean the so called lattice dynamical systems (LDS) or 1D spatiotemporal discrete systems. In [4] one can see the importance of these type of systems.

To analyze when one of these type of systems has a complicated dynamics or not by the observation of one topological dynamical property is an open problem (see [5]). In [5], by using the notion of chaos, the authors characterized the dynamical complexity of a coupled lattice system stated by Kaneko [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction. They proved that this coupled map lattice (CML) system with $f_n = \Lambda$ for every $n \in \{1, 2, \ldots, L\}$ is chaotic in the sense of both Devaney and Li–Yorke for zero coupling constant, where Λ is the tent map. Also, some problems on the dynamics of this CML system with $f_n = \Lambda$ for every $n \in \{1, 2, \ldots, L\}$ were stated by them for the case of having non-zero coupling constants.

Inspired by García Guirao and Lampart [7], we will investigate into the dynamical properties of the following more general LDS:

$$x_n^{m+1} = (1-\varepsilon) f_n\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f_n\left(x_{n-1}^m\right) - f_n\left(x_{n+1}^m\right)\right], \tag{1}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in [0, 1]$ is coupling constant and f_n is a continuous selfmap of I for every $n \in \{1, 2, \ldots, L\}$. In particular, we prove that for zero coupling constant, if there exists $n \in \{1, 2, \ldots, L\}$ such that f_n has positive topological entropy, then this CML system has positive topological entropy. Our result extends the existing one.

2 Preliminaries

Firstly we recall some notations and some concepts. Throughout this paper, X is a compact metric space with metric d, (X, f) is a TDS and $I = [0, 1]$.

A pair of points $x, y \in X$ is called a Li–Yorke pair of system (X, f) if the following conditions are satisfied:

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.
(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

A subset $S \subset X$ is called a LY-scrambled set for f (Li–Yorke set) if the set S has at least two points and every pair of distinct points in S is a Li–Yorke pair. A system (X, f) or a map $f : X \to X$ is said to be chaotic in the sense of Li–Yorke if it has an uncountable scrambled set.

The state space of LDS is the set

$$\mathcal{X} = \left\{x : x = \{x_i\}, x_i \in \mathbb{R}^a, i \in \mathbb{Z}^b, \|x_i\| < \infty\right\}.$$

where $a \geq 1$ is the dimension of the range space of the map of state x_i, $b \geq 1$ is the dimension of the lattice and the l^2 norm

$$\|x\|_2 = \left(\sum_{i \in \mathbb{Z}^b} |x_i|^2\right)^{\frac{1}{2}}$$

is usually taken ($|x_i|$ is the length of the vector x_i) (see [5]).

We will deal with the following more general systems which generalize CML system stated by Kaneko [6] (for more details see for references therein) which is related to the Belusov–Zhabotinskii reaction (for this point we refer to [8], and for experimental study of chemical turbulence by this method one can see [9–11]):

$$x_n^{m+1} = (1 - \varepsilon) f_n\left(x_n^m\right) + \frac{1}{2}\varepsilon\left[f_n\left(x_{n-1}^m\right) - f_n\left(x_{n+1}^m\right)\right], \tag{2}$$

where m is discrete time index, n is lattice side index with system size L, $\varepsilon \in [0, 1]$ is coupling constant and f_n is a continuous selfmap on I for every $1 \leq n \leq L$.

In general, we assume that one of the following periodic boundary conditions of the system (1) or (2) is true:

(1) $x_n^m = x_{n+L}^m$,
(2) $x_n^m = x_n^{m+L}$,
(3) $x_n^m = x_{n+L}^{m+L}$,

standardly, the first case of the boundary conditions is used.

3 Main results

Let d be the product metric on the product space I^L, i.e.,

$$d((x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L)) = \left(\sum_{i=1}^{L} (x_i - y_i)^2\right)^{\frac{1}{2}}$$

for any $(x_1, x_2, \ldots, x_L), (y_1, y_2, \ldots, y_L) \in I^L$.

In mathematics, the topological entropy of a topological dynamical system is a nonnegative real number that measures the complexity of the system. Topological entropy was first introduced in 1965 by Adler et al. [12]. Their definition was modelled after the definition of the Kolmogorov–Sinai, or metric, entropy. Later Dinaburg and Rufus Bowen gave a new, but equivalent definition (see [13,14]). Now, we recall this equivalent definition formulated by Bowen [13], and independently by Dinaburg [14].

Let (X, d) be a metric space and $x \in X$, and let $f : X \to X$ be a uniformly continuous map. For any $n \in \mathbb{N}$ and any $\varepsilon > 0$, a set $E \subset X$ is (n, ε)-separated with respect to f if $x, y \in E$ and $x \neq y$ then $\max\{d(f^i(x), f^i(y)) : 0 \geq i \leq n-1\} > \varepsilon$. For any compact subset $K \subset X$, let $s_n(\varepsilon, K)$ denote the largest cardinality of any (n, ε)-separated subset of K with respect to f. We set $s(\varepsilon, K, f) = \limsup_{n\to\infty} \frac{1}{n} \log s_n(\varepsilon, K)$ for any $\varepsilon > 0$ and any compact subset K of X. The topological entropy of a uniformly continuous map $f : X \to X$ on a metric space X with metric d is a number $h(f) \in [0, +\infty]$ defined by

$$h(f) = \limsup_{K} \lim_{\varepsilon\to 0} s(\varepsilon, K, f),$$

where the supremum is taken over the collection of all compact subsets.

In [7] the authors proved that if $f_n = \Lambda$ for every $n \in \{1, 2, \ldots, L\}$, then system (1) or system (2) has positive topological entropy for zero coupling constant. Inspired by this result we have the following theorem.

Theorem 3.1 *For zero coupling constant, if there is $n \in \{1, 2, \ldots, L\}$ such that f_n has positive topological entropy, then the system (2) has positive topological entropy.*

Proof For $\varepsilon = 0$, it is clear that the system (2) is equivalent to the system $(I^L, f_1 \times f_2 \times \cdots \times f_L)$. From [15] we know that $h(f_1 \times f_2 \times \cdots \times f_L) = h(f_1) + h(f_2) + \cdots + h(f_L)$. Obviously, $h(f_1 \times f_2 \times \cdots \times f_L) \geq h(f_i)$ for every $i \in \{1, 2, \ldots, L\}$. By hypothesis, we have $h(f_1 \times f_2 \times \cdots \times f_L) \geq h(f_n) > 0$. Thus, the proof is finished. □

Remark 3.1 The above theorem extends Theorem 1 in [7].

Example 3.1 Let $f_n = \Lambda^n$ for every $n \in \{1, 2, \ldots, L\}$ and Λ be the tent map. Then the system (2) has positive topological entropy.

For any given coupling constant $\varepsilon \in (0, 1]$, the dynamical behaviour of the system (2) is more complicated. So, we have the following problem.

Problem 3.1 For any coupling constant $\varepsilon \in (0, 1]$, is the above three results true?

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